The demography of the Austrian Academy of Sciences

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Abstract

In a hierarchical organization whose total membership size remains constant the annual intake is strictly determined by the number of deaths and a statutory retirement age. Faced with a rising life expectancy especially for older individuals the average age of a population increases. There is a fundamental dilemma of two conflicting goals of a constant-sized age-structured population, e. g. an Academy of Sciences: to keep a *young* age-structure while to guarantee a *high* recruitment rate. In this paper we first present a reconstruction of the population of the Austrian Academy of Sciences from 1847 to 2005. Based on alternative scenarios of the age distribution of incoming members we project the population of the Austrian Academy forward in time and study the sensitivity of the total number of members, their age distribution and the number of recruits for those alternative scenarios. We conclude our paper by introducing an age-structured optimal control model to determine the optimal trade-off between the rate of replacement and the mean age of a constant-sized population whose dynamics is modelled by the McKendrick partial differential equation. A variant of Pontryagin's maximum principle is derived and used to determine the optimal recruitment distribution in the stationary case. It turns out that due to an U-shaped age-specific shadow price of a member it is optimal to elect either young or old aged new members. We discuss some interesting policy implications of the obtained optimal recruitment policy (scientific excellence and life long achievements).

1 Motivation

The dynamics of small populations is a neglected field in mathematical demography, despite its importance in several branches of science. The development of Learned Societies, universities, armies or other hierarchical organizations can be studied by methods of population dynamics and intertemporal optimization. The purpose of this paper is to show how methods of demography can be applied to *reconstruct* and *project* Learned Societies, in particular the Austrian Academy of Sciences. Moreover we also demonstrate the use of age structured optimal control theory to derive optimal recruitment policies. Since age is one central variable in population dynamics (as seniority is in manpower planning) the appropriate tool of controlling such systems is distributed parameter control (see, e.g., Feichtinger and Hartl 1986, Appendix A.5).

2 Demographic Development of the Austrian Academy of Sciences 1847–2005

Based on biographic records that include

- the date of birth and date of death
- date of election
- information of class membership
- membership status (full vs. corresponding members)
- if applicable year of change in membership status

we reconstructed the population of the Austrian Academy of Sciences (Figure 1) between its foundation in 1847 and 2005.

The population of the Austrian Academy of Sciences is determined by the age distribution of entries, various statutory requirements (cf. Table 1) on e. g. size, rank transitions, etc. and exits that are due to resignation, dismissal and death.

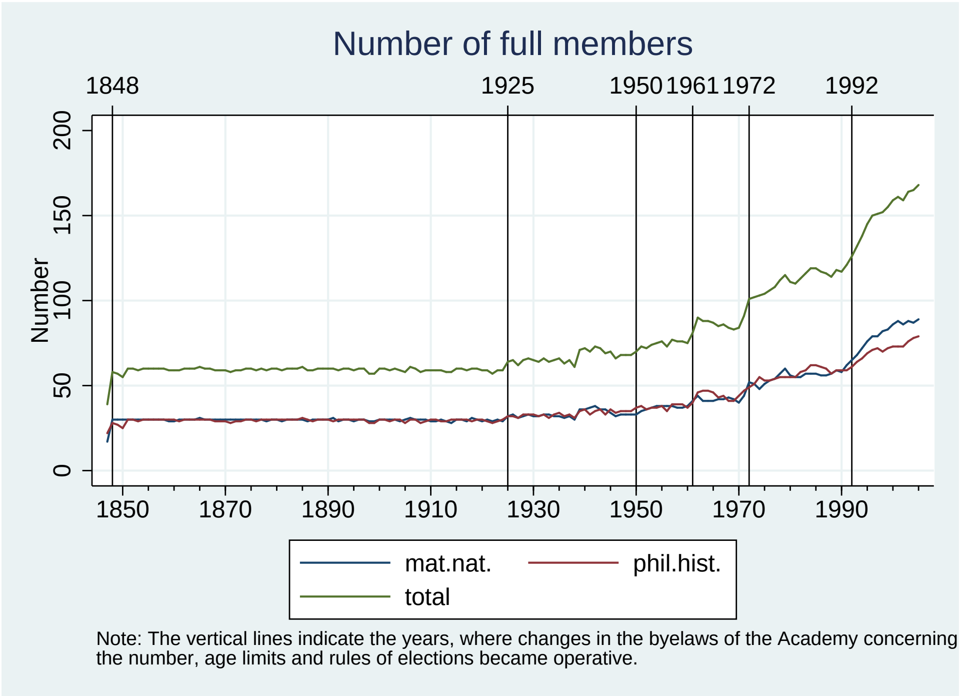

Figure 1: Number of full members by class, 1847–2005.

Table 1: Changes in the byelaws concerning the number of members per class, age limits and rules of elections and the year they became operative.

Byelaws	**Year**[a]	**Members per class**	**Specific rules**
Statuten, 1st version (1847)	1847	24	
- Addendum (1848)	1848	30	
Satzung, 1st version			
- Addendum (1925)	1925	33	
Satzung, 3rd version			
- Addendum (1949)	1950	33	Age limit of 75; at maximum five new members elected per year.
- Addendum (1960)	1961	33	Restriction of elections dropped.
- Addendum (1971)	1972	33	Age limit of 70.
- Addendum (1991)	1992	45[b]	

[a] Year of election, where the change became effective first.
[b] Informal agreement to distribute additional elections over three years.

The rapid decrease of mortality particularly in older ages has brought about an increasing ageing of Academies. In Figure 2 we plot the mean age of full members for both classes.

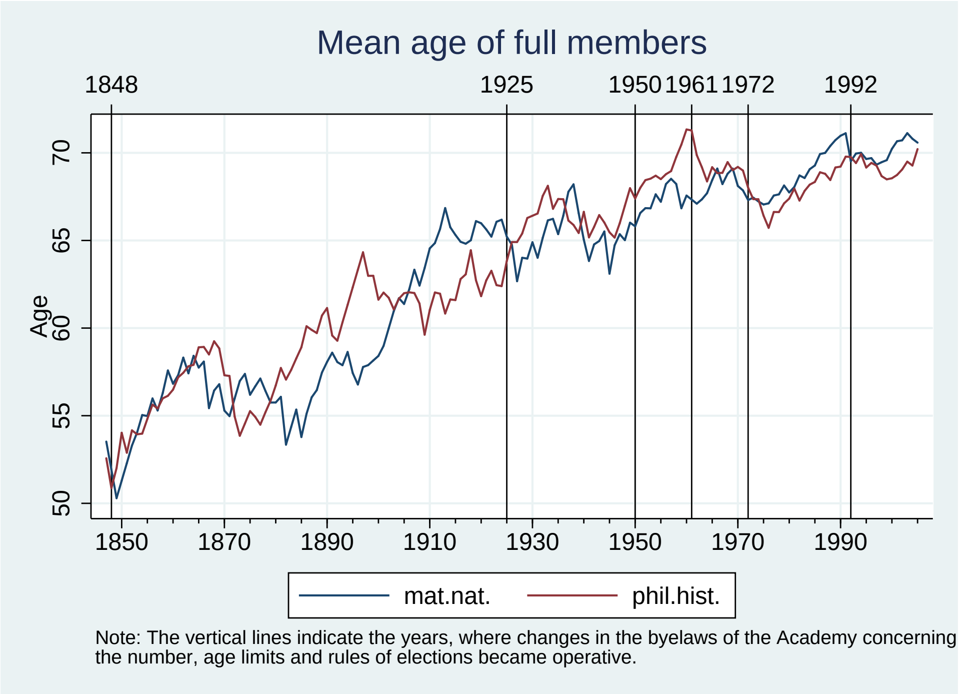

Figure 2: Mean age of full members by class, 1847–2005.

Moreover, the mean age at election increased as well over time (see Figure 3), which further contributed to the ageing of the Academy.

In addition, the members of the Austrian Academy of Sciences exhibit a lower mortality than the average Austrian population. Figure 4 shows the standardized mortality ratio compared to Austrian life table male mortality over time in 10-year periods starting from 1866/75. The standardized mortality ratio gives the ratio of observed to expected deaths, if the members of the Austrian Academy of Sciences would be subject to reference death rates (in our case the Austrian male life table death rates). As evident from Figure 4, the SMR is always below 1, except of the first period, although not statistically significantly different from unity for the periods 1876–1885, 1896–1905, 1926–1935, and 1956–1965 as verified by a score test (Clayton and Hills 1993). Moreover, we find that from the late 19th century to the mid of the 20th century, the standardized mortality ratio was around 0.75 and fell to about 0.5 in the second half of the 20th century.

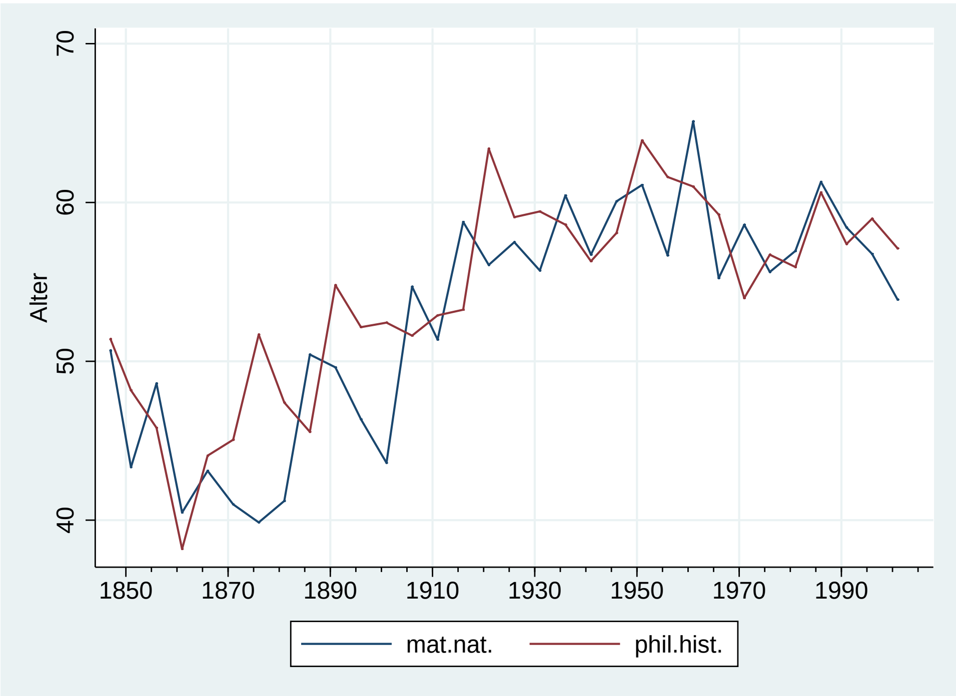

Figure 3: Mean age of full members at election by class (5-year periods), 1847–2005.

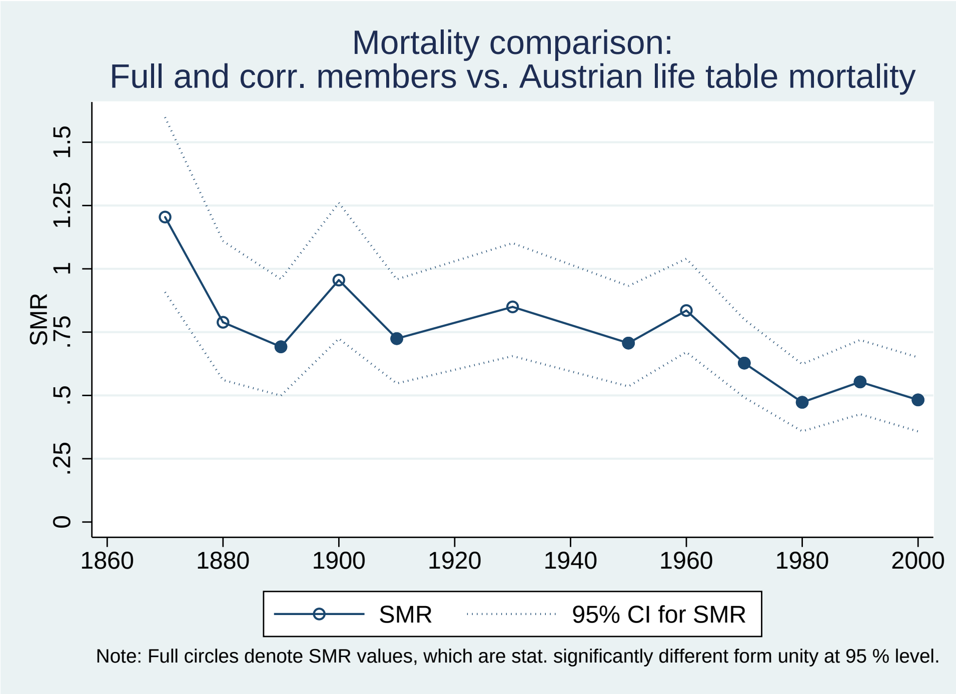

Figure 4: Standardized mortality ratio of members compared to Austrian life table mortality over time.

3 Projections

Over-ageing of professional organizations or bodies has been frequently seen as a disadvantage. An Academy of Sciences as an advisory body should stay in touch with the community of working researchers. During the last decades several new important branches of sciences, e.g. in informatics, biology, ecology, etc., developed which should be represented by young dynamic scientists. To reach a rejuvenation of the Academy, there exist three measures:

1. raising the number of members;
2. limiting the population of members by an upper age limit (70 years in the Austrian Academy of Sciences);
3. to recruit young members.

The first "tool" has its pendant in the dynamics of large human populations which are governed by the alternative "to age or to grow". However, for various learned societies (as the Academie des Sciences in France or the Austrian Academy of Sciences) this remedy is excluded due to a constant membership below an limit age (150 members below 75 in France, 90 full members below 70 in Austria). As just mentioned the second instrument is frequently used.

This leaves the third remedy as the essential steering possibility to be applied at the annual elections. There is, however, a fundamental dilemma in rejuvenating an age-structured population with constant size. Let us clarify it be a simple example which has only benchmark character.

If the Academy recruits only 47,5 year old new members, they stay 22,5 years in the system (neglecting mortality until 70 and other possibilities for exit) until the statutory retirement age. The Austrian Academy has 90 full members (45 in each class) yielding $90 : 22{,}5 = 4$ new entrants each year. If, on the other hand, only 55 year old entrants are recruited, the same calculation delivers $90 : 15 = 6$ persons.

This means that the younger the age at election, the longer the tenure in the Academy and the lower is the rate of intake. Note that a younger recruitment distribution means a younger age structure of the members measured, e.g., by the average age of the population.

It is clear that a flourishing Academy wants to recruit as many new members as possible. As already mentioned, there are new important disciplines which should be well represented in the Academy. This amounts to the central question posed in the subsequent simulations and mathematical formalization, to find an optimal trade-off between the two conflicting goals, of the Academy namely a young age structure and as many entrants as possible. This fundamental trade-off of a constant size population may be illustrated by a hyperbola as in Figure 5.

Denoting by M the total size of the body, R the member of annual intakes, and T the mean length of tenure, the stationary state is characterized by the relation

$$M = RT \tag{1}$$

In the Austrian case we have $M = 90$, $0 \leq T \leq 30$ (assuming a minimal age at entry of 40 years and an upper age limit of 70).

Remark 1: Note that the relation (1) is fundamental in a stationary population where the stock equals the number of births times the life expectancy. In queuing theory which is based on birth-death processes (1) is well-know as Little's formula (see Hillier and Lieberman (1974), p.384).

Note that the dilemma has been formulated in the important contribution of Leridon (2004) describing the demography of the French Academy of Sciences. He puts it in that way: "To counteract the spontaneous trends in ageing in the institution new members would have to be elected at increasingly young ages year after year, which would have the drawback of reducing the rate of population replacement".

To study the implication of the age distribution of new entries for the number and structure of the members of the Austrian Academy of Sciences

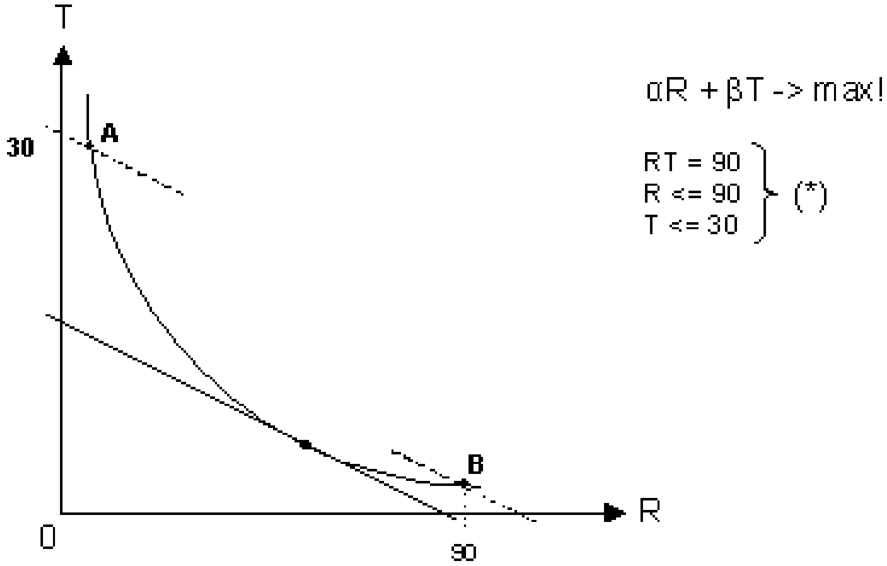

Figure 5: Trade-off between recruitment R and average tenure T

we apply demographic projection methods. We consider four alternatives scenarios of the age distribution of new entries (see Figure 6)

Status quo: Average age distribution observed during the last 25 years.

Young: Only persons below age 55 are elected.

Old: Only persons above age 55 are elected.

Bimodal: Persons between age 40 to 49 and 60 to 69 are elected.

The resulting number of members, number of vacancies and population that is below age 70 are plotted in Figures 7 to 9. As these figures indicate, the dilemma between a "young" society that only allows a small number of vacancies each year and a small number of the total stock of members vs. an "old" society that allows more vacancies each year and a larger stock of members each year becomes obvious. The bimodal distribution of the age at entry seems to be a compromise and as we will show in the subsequent section it also constitutes an optimal recruitment policy if the Academy's goal is to keep its member structure young and to allow a maximum number of recruitments each period.

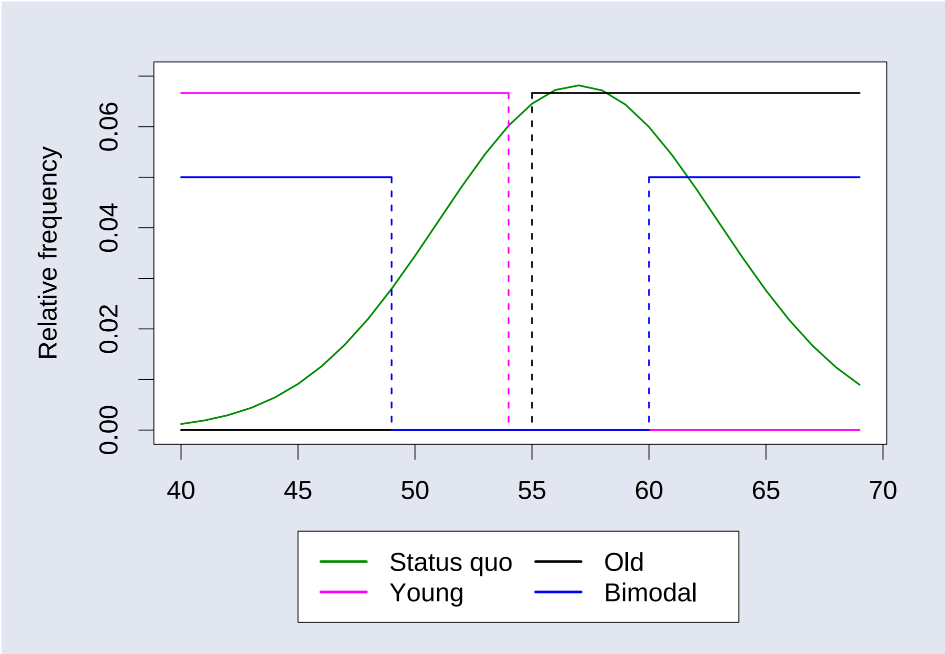

Figure 6: Projection scenarios about the age distribution of election.

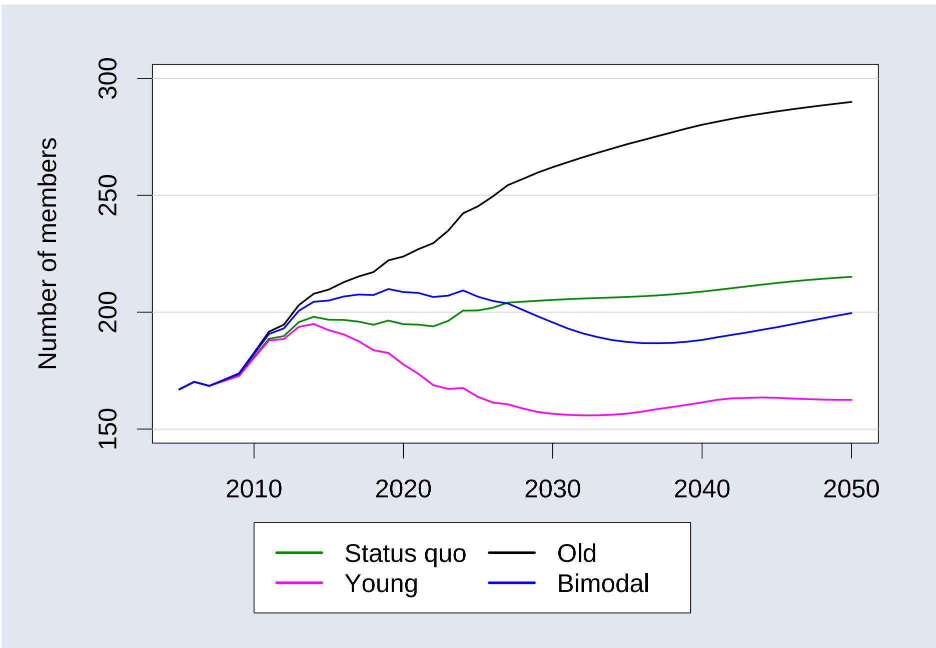

Figure 7: Projected number of members (both classes together).

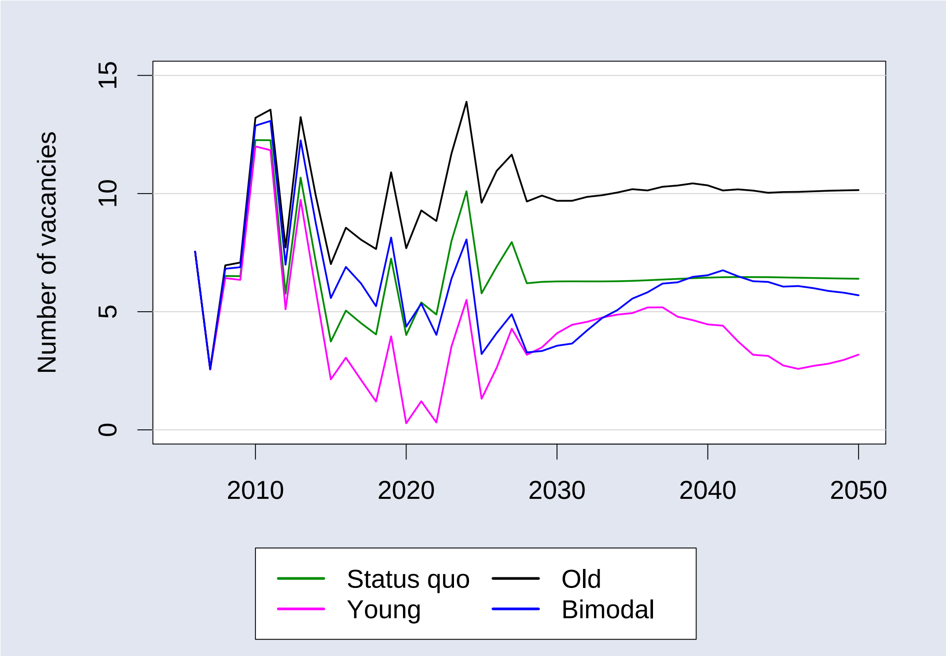

Figure 8: Projected number of vacancies (both classes together).

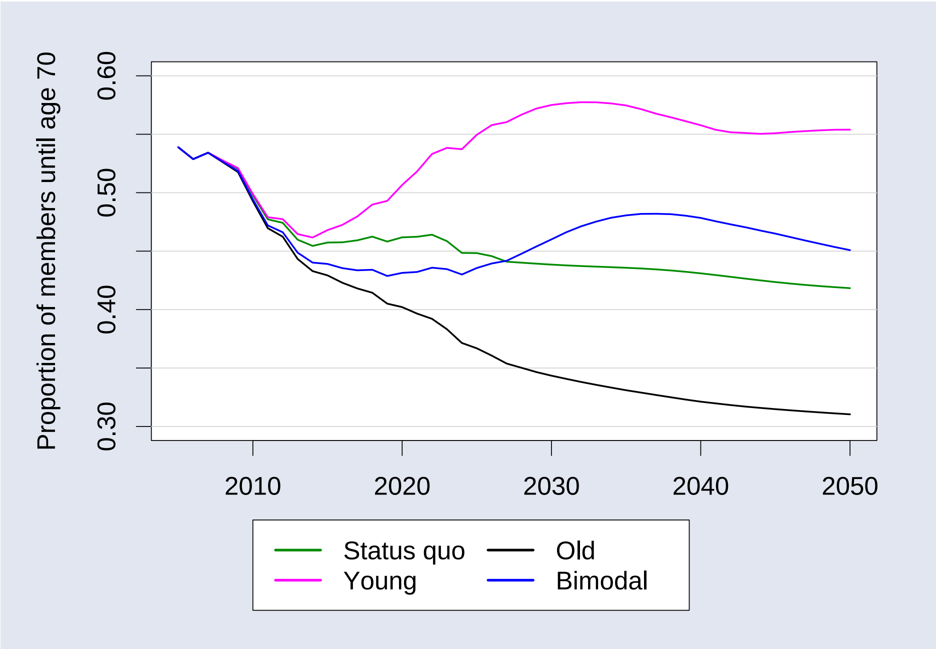

Figure 9: Projected proportion of members aged less or equal 70 (both classes together).

4 Optimal Age-structured recruitment when a fixed-size organization should be young

The following variables will be involved in the model below. $t \geq 0$ is time, $a \in [\underline{a}, \bar{a}]$ is age,
$\underline{a}$ and $\bar{a}$ are the minimal and the maximal age of membership, respectively[1],
$M(t, a)$ is the number of members of the academy at time t of age a,
$\bar{M}$ is the total number of members, which is supposed to be constant in time.

4.1 First formulation

Changes in the composition of the Academy occur due to mortality (at a time and age dependent rate $\mu(t, a)$) due to retirement at age $\bar{a}$, and due to recruitment, which happens with an age-dependent intensity $r(t, a)$. The classical McKendrick equation holds:

$$M_t + M_a = -\mu(t, a)M(t, a) + r(t, a),$$

with initial and boundary conditions

$$\begin{aligned} M(0, a) &= M_0(a) \quad \text{– a given initial density} \\ M(t, \underline{a}) &= 0. \end{aligned}$$

The objective is to maximize

$$\int_0^\infty e^{-\rho t} \left[\alpha R(t) - \beta \int_{\underline{a}}^{\bar{a}} A(a, M(t, a), R(t), r(t, a)) \, \mathrm{d}a \right] \mathrm{d}t,$$

where

$$R(t) = \int_{\underline{a}}^{\bar{a}} r(t, a) \, \mathrm{d}a,$$

is the recruitment trajectory and α and β are non-negative weights for the amount of recruitment and for the age structure[2], respectively, and $\rho \geq 0$ is a discount rate. Maximization is done under the following constraints

$$r(t, a) \geq 0,$$

[1]For the Austrian Academy $\underline{a} = 40$ (leaving aside very few exception) $\bar{a} = 70$ is the age limit according to the Academy rules from 1972

[2]Notice that the two terms in the objective function are measured in different units. The first is measured in people/time. If the meaning of A is average age (the first case below) the second term is measured by units of time. If a decision maker chooses values $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$ this means that he/she considers as equally desirable increasing the number of recruitments by $\bar{\alpha}$ and decreasing the average age by $\bar{\beta}$. Certainly one can normalize the values of α and β so that $\alpha + \beta = 1$.

and since the total number of members is fixed, the policy r should ensure

$$\int_{\underline{a}}^{\bar{a}} M(t,a)\,\mathrm{d}a = \bar{M}.$$

For the function $A(a, M, R, r)$ in the objective there are different possibilities

1. Average age of the members
$$A(a, M, D, d) = \frac{aM}{\bar{M}}.$$

2. Average age at recruitmemt
$$A(a, M, D, d) = \frac{ar}{R}.$$

3. Deviation from a wished age distribution of the members $\eta^M(a)$
$$A(a, M, R, r) = \left(\frac{M}{\bar{M}} - \eta^M(a)\right)^2.$$

4. Deviation from a wished age distribution of recruitment $\eta^r(a)$
$$A(a, M, R, r) = \left(\frac{r}{R} - \eta^d(a)\right)^2.$$

The above problem is technically complicated by the presence of the state constraint requiring that the size of the Academy is fixed. Therefore we pass to another formulation in the next section.

4.2 Reformulation

Substitute
$$r(t,a) = R(t)u(t,a)$$
with $\int_{\underline{a}}^{\bar{a}} u(t,a)\,\mathrm{d}a = 1$ and $u(a) \geq 0$. Thus $R(t)$ is the magnitude, and $u(t,\cdot)$ is the age-density of the hiring intensity.

Integrating the state equation with respect to a leads to

$$\dot{\bar{M}} + M(t,\bar{a}) - M(t,\underline{a}) = -\int_{\underline{a}}^{\bar{a}} \mu(t,a)M(t,a)\,\mathrm{d}a + R(t).$$

Since $\dot{\bar{M}} = 0$ and $M(t,\underline{a}) = 0$, we get

$$R(t) = M(t,\bar{a}) + \int \mu(t,a)M(t,a),$$

which ensures the state constraint $\int_{\underline{a}}^{\bar{a}} M(t,a)\,\mathrm{d}a = \bar{M}$.

Moreover, we change the age variable substituting $a := a - \underline{a}$ and denote $\omega = \bar{a} - \underline{a}$. Overloading somewhat the notation A we obtain the following problem:

$$\max \int_0^\infty e^{-\rho t} \left[\alpha R(t) - \int_0^\omega \beta A(a, M(t,a), u(t,a))\,\mathrm{d}a \right] \mathrm{d}t \tag{2}$$

subject to

$$M_t + M_a = -\mu(t,a)M(t,a) + R(t)u(t,a), \tag{3}$$

$$M(0,a) = M_0(a), \quad M(t,0) = 0, \tag{4}$$

$$R(t) = M(t,\omega) + \int_0^\omega \mu(t,a)M(t,a)\,\mathrm{d}a, \tag{5}$$

and the additional constraints

$$0 \le u(t,a) \le \bar{u}(a), \qquad \int_0^\omega u(t,a)\,\mathrm{d}a = 1. \tag{6}$$

The upper bound, $\bar{u}(a)$, for $u(t,a)$ is introduced in order to avoid solutions with very high or infinite density $u(a)$ in some ages. One can take $\bar{u} = +\infty$ for the forth choice of the function A. The four proposed choices of A become now

$$A(a,M,u) = \frac{a}{\bar{M}} M, \tag{7}$$

$$A(a,M,u) = au, \tag{8}$$

$$A(a,M,u) = \left(\frac{M}{\bar{M}} - \eta^M(a) \right)^2,$$

$$A(a,M,u) = (u - \eta^r(a))^2.$$

5 Maximum principle

We introduce the adjoint system

$$\xi_t + \xi_a = (r + \mu(t,a))\xi - \mu(t,a)\eta(t) - \alpha\mu(t,a) + \beta A_M,$$

$$\eta(t) = \int_0^\omega \xi(t,a)u(t,a)\,\mathrm{d}a,$$

with side conditions

$$\xi(t,\omega) = \alpha + \eta(t), \quad \limsup_{t\to+\infty} \sup_{a\in[0,\omega]} \xi(t,a) < +\infty. \tag{9}$$

Theorem 1 *Assume that μ is measurable and bounded, A is differentiable in M with locally Lipschitz derivative and $\rho > 0$*[3]*. Let (u, M, R) be an optimal solution and let u be bounded. Then the above adjoint system has an unique solution*[4] *$\xi(t, a)$ and the optimal control $u(t, \cdot)$ maximizes for every t the integral*

$$\int_0^\omega [\xi(t,a)R(t)u(a) - \beta A(a, M(t,a), u)] \, \mathrm{d}a$$

on the set of functions $u(\cdot)$ satisfying (6).

Since the problem above involves a sort of aftereffect (due to $M(t, \omega)$ in the right-hand side of the differential equation, the claim of the above theorem does not follow from Feichtinger et al. (2003) or from other results concerning McKendrick type control systems. Therefore we present a proof in Appendix.

Remark 1 Since, as usual, $\xi(t, a)$ has the maening of a "shadow price" of the members of age a at time t, the quantity $\eta(t)$ is the "shadow price" of the reqruited members at time t. The adjoint equation and the boundary conditions can then be interpreted in the terms of the "shadow price".

For further anlytical results we refer to the paper by Feichtinger and Veliov (2005) and only present numerical solutions in the following.

5.1 Numerical solutions for the Austrian Academy of Sciences

Here we have $\bar{M} = 90$, $\omega = 30$.

Figures 10–12 present the optimal shadow price and optimal stock of members for three sets of parameters α and β: $(0.6, 0.4)$, $(0.5, 0.5)$, and $(0.1, 0.9)$, respectively. Here the constraint $\bar{u}(a) = 0.077$[5] is independent of the age. With this choice of $\bar{u}$ recruitment occurs in about 43% of the age horizon $[0, \omega]$, namely, $\theta + \tau = 0.435\omega$.

Now we consider two examples with non-constant bound $\bar{u}(a)$. In both cases $\alpha = \beta = 0.5$.

In the first example (Figures 13 and 14) $\bar{u}(a)$ decreases linearly from the value $\bar{u}(0) = 0.12$ to $\bar{u}(25) = 0$, then equals zero on $[25, 30]$.

[3] An easier alternative to consider the case $\rho = 0$ on a finite horizon $[0, T]$, in which case the transversality condition below should be replaced with $\xi(T, a) = 0$.

[4] The exact meaning of a solution to the primal and the adjoint system is similar to that in Feichtinger et al. (2003).

[5] This value means that the recruitments in every one of the 30 age groups (each one consisting of candidates born in the same year) does not exceed 7.7% of all reqruitments. In particular, each year there must be recruitments in at least (in fact, exactly) $100 : 7.7 \approx 13$ age groups.

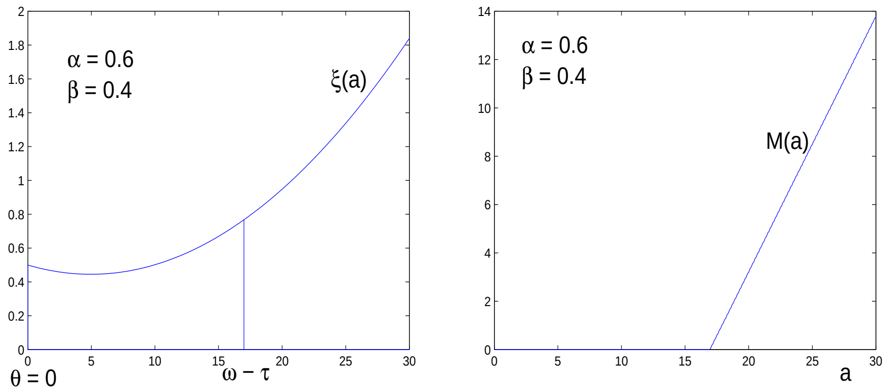

Figure 10: The adjoint variable ξ and the optimal switching ages (left) and the optimal $M(a)$ (right) for a constant bound $\bar{u}(a) = \bar{u}$.

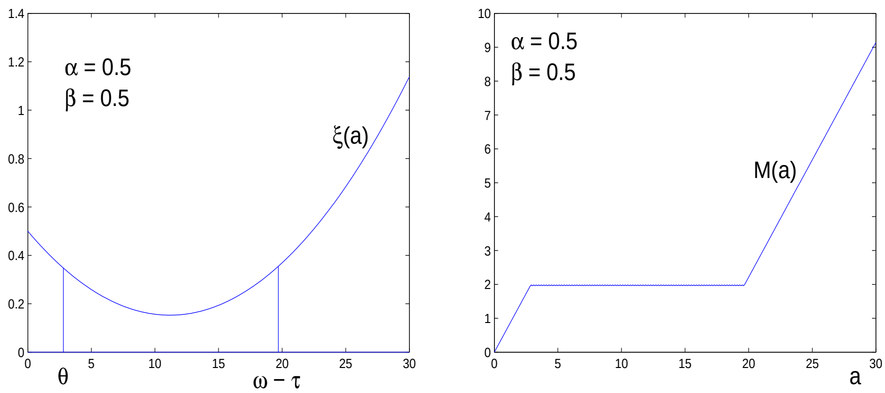

Figure 11: The adjoint variable ξ and the optimal switching ages (left) and the optimal $M(a)$ (right) for a constant bound $\bar{u}(a) = \bar{u}$.

In the second example (Fig. 15 and 16) $\bar{u}(a)$ linearly increases from zero to the value $\bar{u}(12.5) = 0.16$, then linearly decreases from this value to $\bar{u}(25) = 0$, and equals zero on $[25, 30]$.

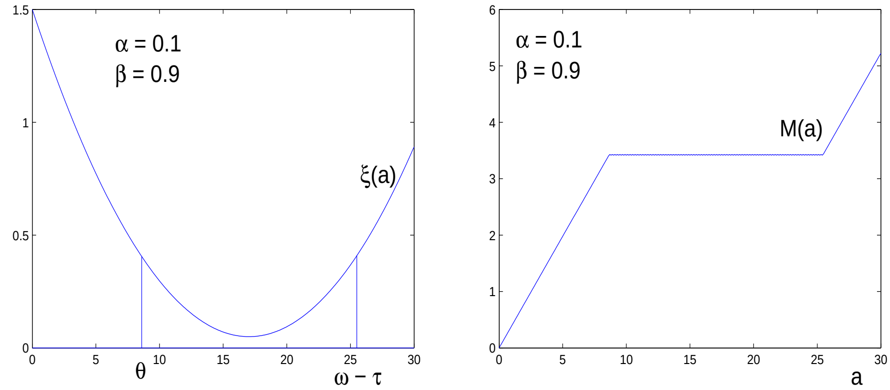

Figure 12: The adjoint variable ξ and the optimal switching ages (left) and the optimal $M(a)$ (right) for a constant bound $\bar{u}(a) = \bar{u}$.

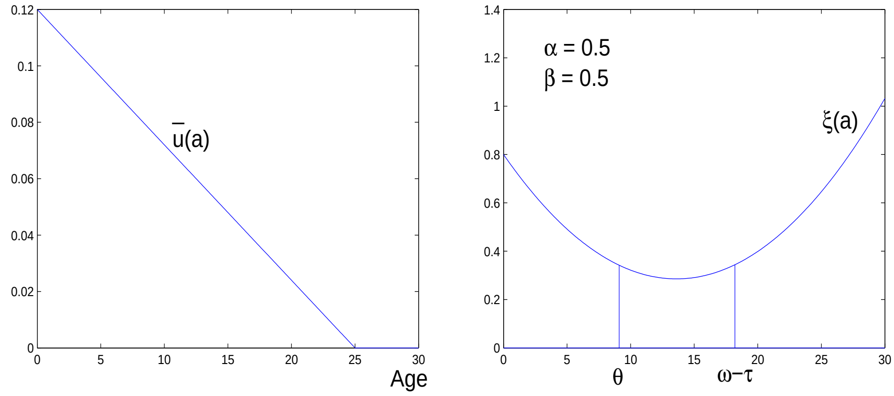

Figure 13: The graphs of $\bar{u}$, the optimal ξ.

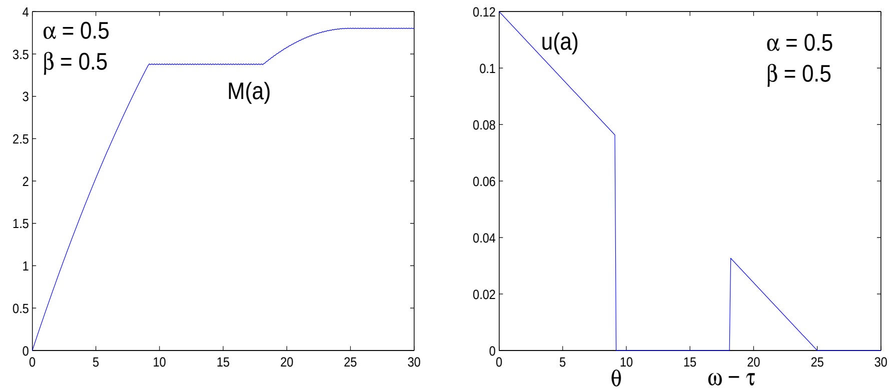

Figure 14: The graphs of the optimal M and u.

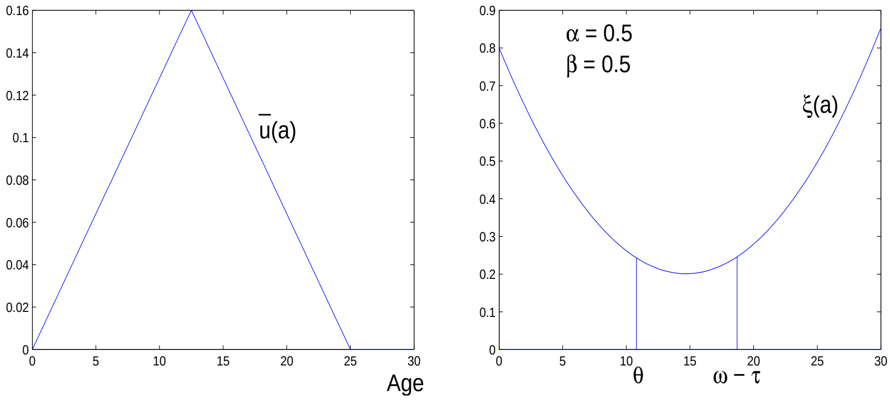

Figure 15: The graphs of $\bar{u}$, the optimal ξ.

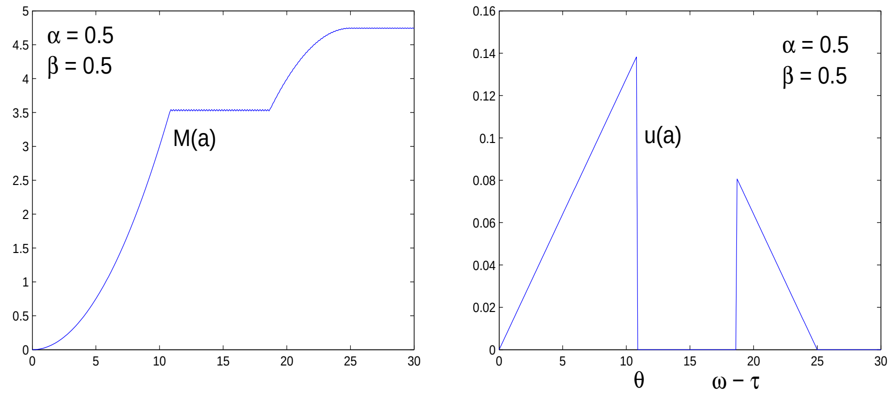

Figure 16: The graphs of the optimal M and u.

References

Clayton, David and Michael Hills. 1993. *Statistical Models in Epidemiology.* New York: Oxford University Press.

Feichtinger, G. and V.M. Veliov. 2005. Optimal age-structured recruitment when a fixed-size organization should be young. *Working Paper.*

Feichtinger, Gustav and Richard F. Hartl. 1986. *Optimale Kontrolle ökonmischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften.* Berlin: De Gruyter.

Hillier, GF. and G. Lieberman. 1974. *Operations Research, 2-nd edition.* San Francisco: Holden Day, Inc.

Leridon, H. 2004. The demography of a learned society. academie de sciences (institut de france), 1666-2030. *Population - E 59(1)* 81–114.